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On an Interpolation Process of S. N. Bernstein

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1. INTRODUCTION

Let $f(x)$ be an arbitrary real continuous function on $[-1, +1]$ and let us denote

$$N = n+1, \qquad U_n(x) = \frac{\sin N\theta}{\sin \theta}, \qquad x = \cos\theta; \qquad n = 1, 2, \ldots \tag{1.1}$$

$U_n(x)$ is known as the Chebyshev polynomial of the second kind of degree n. $U_n(x)$ has all its zeros at the points

$$x_k = \cos\theta_k, \qquad \theta_k = k\pi/N, \qquad k = 1, 2, \ldots, n. \tag{1.2}$$

Let

$$\mu_k(x) = \frac{(-1)^{k+1}(1-x_k^2)}{N} \cdot \frac{U_n(x)}{x - x_k}, \qquad k = 1, 2, \ldots, n, \tag{1.3}$$

the fundamental polynomials of Lagrange interpolation based on the nodes x_k. A. K. Varma (1) considered the following interpolation process based on the nodes (1.2):

$$F_n(f, x) = \sum_{k=1}^{n} f(x_k)\, m_k(x), \qquad n \geqslant 3, \tag{1.4}$$

where

$$m_1(x) = \frac{3\mu_1(x) + \mu_2(x)}{4},$$
$$m_k(x) = \frac{\mu_{k+1}(x) + 2\mu_k(x) + \mu_{k-1}(x)}{4}, \qquad k = 2, 3, \ldots, n-1, \tag{1.5}$$
$$m_n(x) = \frac{\mu_{n-1}(x) + 3\mu_n(x)}{4}.$$

Concerning $F_n(f, x)$ he proved the following

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THEOREM (Varma). *Let $f(x)$ be an arbitrary continuous real function on $[-1, +1]$ and let $\omega(\delta)$ be its modulus of continuity there. Then there exists a positive constant c, independent of n, x, and f, such that*

$$\max_{-1 \leqslant x \leqslant 1} |F_n(f, x) - f(x)| \leqslant c\omega(1/n). \tag{1.6}$$

From (1.6) it follows that, generally speaking, convergence order reaches best approximation order when the interpolation process $F_n(f, x)$ is used to approximate a real continuous function on $[-1, +1]$. It is natural to raise this very interesting question: If $f(x)$ has a continuous derivative on $[-1, +1]$, then when, using the interpolation process $F_n(f, x)$ to approximate $f(x)$ on $[-1, +1]$, can the convergence order reach best approximation order? We give here the following definite answer:

THEOREM. *Let $f(x)$ be an arbitrary real function $\in C^1[-1, 1]$. Then*

$$|F_n(f, x) - f(x)| = O\left(\frac{1}{n}\right) \omega\left(f', \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) + O\left(\frac{\|f'\|}{n^2}\right),$$
$$-1 \leqslant x \leqslant 1, \tag{1.7}$$

where O is independent of n and f, $\|f'\| = \max_{-1 \leqslant x \leqslant 1} |f'(x)|$, and $\omega(f', \delta)$ is the modulus of continuity of f' on $[-1, 1]$.

From (1.7) we see that, using $F_n(f, x)$ to approximate $f(x) \in C^1$ on $[-1, +1]$, the convergence order reaches that of best approximation. Further, the last theorem yields the

COROLLARY. *On $[-1, 1]$, let $f'(x) \in \operatorname{Lip} \alpha$, $0 < \alpha \leqslant 1$. Then, by the subadditivity of ω,*

$$|F_n(f, x) - f(x)| = O(1) \left[\frac{(\sqrt{1-x^2})^\alpha}{n^{1+\alpha}} + \frac{1}{n^{1+2\alpha}} + \frac{\|f'\|}{n^2}\right], \qquad -1 \leqslant x \leqslant 1.$$

2. SOME LEMMAS

To prove the last theorem, we need the following lemmas.

LEMMA 1 (Cf. [5, 6]). *Let x_j be the nearest node to $x \in [-1, 1]$. Then*

$$|f'(X) - f'(x)| = O(1)\, \omega\left(f', \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \qquad k = j-1, j, j+1. \tag{2.1}$$

If $1<k<j$ *or* $j<k<n$, *then*

$$|f'(X)-f'(x)|=O(1)\,\omega\left(f', \frac{|k-j|\sqrt{1-x^2}}{n}+\frac{(k-j)^2}{n^2}\right), \quad (2.2)$$

$$|f'(t_k)-f'(t_{k+1})|=O(1)\,\omega\left(f', \frac{\sqrt{1-x^2}}{n}+\frac{|k-j|}{n^2}\right), \quad (2.3)$$

where $x_k \leqslant t_k \leqslant x_{k-1}$, $X=x+a_k(x_k-x)$ $(0\leqslant a_k \leqslant 1)$, *or* $X=t_k$.

LEMMA 2. *The following estimates are valid for* $-1\leqslant x\leqslant +1$, $n\geqslant 5$:

$$m_k(x)=O(1), \qquad k=1, 2, ..., n; \quad (2.4)$$

$$m_k(x)=O(1)\left(\frac{|\sin N\theta|}{n^3\left|\prod_{j=k-1}^{k+1}\sin\frac{\theta-\theta_j}{2}\right|}+\frac{1}{n^2}\right), \qquad k=2, 3, ..., n-1; \quad (2.5)$$

$$p_k(x)=O(1)\left(\frac{|\sin N\theta|}{n^5\left|\prod_{j=k-2}^{k+2}\sin\frac{\theta-\theta_j}{2}\right|}+\frac{1}{n^4}\right), \qquad k=3, 4, ..., n-2; \quad (2.6)$$

where $p_k(x)=m_{k+1}(x)+2m_k(x)+m_{k-1}(x)$.

Proof. For the fundamental polynomials $\mu_k(x)$ we have

$$\mu_k(x)=O(1), \qquad k=1, 2, ..., n; \quad (2.7)$$

thus (2.4) is valid. By [1] we have

$$\mu_k(x)=\frac{(-1)^{k+1}(1-x^2)\,U_n(x)}{N(x-x_k)}+\frac{(-1)^{k+1}(x+x_k)\,U_n(x)}{N},$$

$$k=1, 2, .., n; \quad (2.8)$$

$$\left|\sin\frac{\theta-\theta_k}{2}\right|\leqslant\left|\sin\frac{\theta+\theta_k}{2}\right|; \quad (2.9)$$

$$4m_k(x)=\frac{(-1)^{k-1}\sin^2\frac{\pi}{2N}\sin N\theta}{N}$$

$$\times\left(\frac{\cot\frac{\theta+\theta_k}{2}}{\sin\frac{\theta+\theta_{k+1}}{2}\sin\frac{\theta+\theta_{k-1}}{2}}+\frac{\cot\frac{\theta-\theta_k}{2}}{\sin\frac{\theta-\theta_{k+1}}{2}\sin\frac{\theta-\theta_{k-1}}{2}}\right)$$

$$+\frac{(-1)^{k-1}4\cos\theta_k\sin^2\frac{\pi}{2N}U_n(x)}{N},$$

$$k=2,3,...,n-1. \qquad (2.10)$$

Therefore, using (2.9) and the inequalities

$$(2/\pi)t\leqslant\sin t\leqslant t, \qquad 0\leqslant t\leqslant\pi/2, \qquad (2.11)$$

$$|U_n(x)|\leqslant N, \qquad -1\leqslant x\leqslant 1 \qquad (2.12)$$

the formula (2.5) follows. To prove (2.6) let us denote

$$R_k(\theta)=\frac{N_k(\theta)}{M_k(\theta)}=\frac{\cos\left(\theta_k+\theta+\frac{\pi}{2N}\right)+2\cos\frac{\pi}{2N}}{\prod_{j=k-1}^{k+2}\sin\frac{\theta_j+\theta}{2}}.$$

Then

$$R_{k-1}(\theta)-R_k(\theta)$$

$$=\frac{N_{k-1}(\theta)\left(\sin\frac{\theta+\theta_{k+2}}{2}-\sin\frac{\theta+\theta_{k-2}}{2}\right)-(N_k(\theta)-N_{k-1}(\theta))\sin\frac{\theta+\theta_{k-2}}{2}}{\prod_{j=k-2}^{k+2}\sin\frac{\theta+\theta_j}{2}}$$

$$=\frac{2N_{k-1}(\theta)\sin\frac{\pi}{N}\cos\frac{\theta+\theta_k}{2}+2\sin(\theta+\theta_k)\sin\frac{\pi}{2N}\sin\frac{\theta+\theta_{k-2}}{2}}{\prod_{j=k-2}^{k+2}\sin\frac{\theta+\theta_j}{2}}$$

$$=O\left(\frac{1}{n\left|\prod_{j=k-2}^{k+2}\sin\frac{\theta+\theta_j}{2}\right|}\right).$$

Similarly, we have

$$R_{k-1}(-\theta)-R_k(-\theta)=O\left(\frac{1}{n\left|\prod_{j=k-2}^{k+2}\sin\frac{\theta-\theta_j}{2}\right|}\right).$$

By [1] we have

$$4(m_k(x)+m_{k-1}(x))=\frac{8(-1)^{k-1}\sin^3\frac{\pi}{2N}U_n(x)\sin\frac{(2k+1)\pi}{2N}}{N}$$
$$+\frac{(-1)^{k+1}\sin^3\frac{\pi}{2N}\sin N\theta}{N}(R_k(\theta)-R_k(-\theta)).$$

Thus, using (2.9), (2.11), and (2.12), we get

$$4\,p_k(x)=\frac{8(-1)^k\sin^3\frac{\pi}{2N}U_n(x)}{N}\left(\sin\frac{(2k-1)\pi}{2N}-\sin\frac{(2k+1)\pi}{2N}\right)$$
$$+\frac{(-1)^k\sin^3\frac{\pi}{2N}\sin N\theta}{N}[(R_{k-1}(\theta)-R_k(\theta))$$
$$-(R_{k-1}(-\theta)-R_k(-\theta))]$$
$$=O\left(\frac{|\sin N\theta|}{n^5\left|\prod_{j=k-2}^{k+2}\sin\frac{\theta-\theta_j}{2}\right|}+\frac{1}{n^4}\right),\qquad k=3,4,...,n-2.$$

This proves (2.6), and Lemma 2 is established.

LEMMA 3. *The following estimate is valid for* $-1\leqslant x\leqslant +1$, $n\geqslant 5$:

$$W=\sum_{k=3}^{n-2}\sum_{j=0}^{2}\binom{2}{j}(x_{k+j-1}-x)\,m_{k+j-1}(x)=O\left(\frac{1}{n^2}\right).$$

Proof. Let us denote

$$\eta_k(x)=\frac{(-1)^{k+1}(1-x^2)\,U_n(x)}{N(x-x_k)},\qquad k=1,2,...,n; \tag{2.13}$$

$$\varphi_k(x)=\frac{\eta_{k+1}(x)+2\eta_k(x)+\eta_{k-1}(x)}{4},\qquad k=3,4,...,n-2; \tag{2.14}$$

$$\psi_k(x)=\frac{(-1)^{k-1}\cos\theta_k\sin^2(\pi/2N)\,U_n(x)}{N},\qquad k=3,4,...,n-2. \tag{2.15}$$

Then from (2.8) and (2.10) we have

$$m_k(x)=\varphi_k(x)+\psi_k(x),\qquad k=3,4,...,n-2.$$

Thus

$$\begin{aligned}W&=\sum_{k=3}^{n-2}\sum_{j=0}^{2}\binom{2}{j}(\chi_{k+j-1}-x)\,\varphi_{k+j-1}(x)\\&\quad+\sum_{k=3}^{n-2}\sum_{j=0}^{2}\binom{2}{j}(x_{k+j-1}-x)\,\psi_{k+j-1}(x)\\&=I_1+I_2.\end{aligned}$$

As $x_k=\cos(k\pi/N)$, we have

$$\begin{aligned}&x_{k+1}-2x_k+x_{k-1}=-4\sin^2\frac{\pi}{2N}\cos\theta_k=O\left(\frac{1}{n^2}\right),\\&x_{k+1}^2-2x_k^2+x_{k-1}^2=O\left(\frac{1}{n^2}\right).\end{aligned} \tag{2.16}$$

Thus, on using (2.11)–(2.12) and (2.15), we obtain

$$\begin{aligned}I_2&=\frac{\sin^2(\pi/2N)\,U_n(x)}{N}\sum_{k=3}^{n-2}(-1)^k[(x_{k+1}-x)\,x_{k+1}-2(x_k-x)\,x_k\\&\quad+(x_{k-1}-x)x_{k-1}]=O\left(\frac{1}{n^3}\right).\end{aligned}$$

To estimate I_1 let us denote

$$\Phi_k(x)=\sum_{j=0}^{2}\binom{2}{j}(x_{k+j-1}-x)\,\eta_{k+j-1}(x).$$

Then I_1 can be split into three parts as follows:

$$
\begin{aligned}
4I_1 = & \left\{4\sum_{k=4}^{n-3} + \sum_{k=2,3,n-2,n-1} \beta_k\right\} \Phi_k(x) \\
& + \left\{\sum_{k=2,3} \sum_{j=0}^{2} \binom{2}{j} (x_{k+j} - x_{k+j-1})\, \eta_{k+j-1}(x)\right. \\
& \left. + \sum_{k=n-2,n-1} \sum_{j=0}^{2} \binom{2}{j} (x_{k+j-2} - x_{k+j-1})\, \eta_{k+j-1}(x)\right\} \\
& + \sum_{k=4}^{n-3} [4W_k \varphi_k(x) + (W_{k+1} - W_k)\, \eta_{k+1}(x) + (W_{k-1} - W_k)\, \eta_{k-1}(x)] \\
= & B_1 + B_2 + B_3,
\end{aligned}
$$

where $W_k = x_{k+1} - 2x_k + x_{k-1}$, $\beta_2 = \beta_{n-1} = 1$, $\beta_3 = \beta_{n-2} = 3$. From (2.13) we have

$$\Phi_k(x) = 0, \qquad k = 2, 3, ..., n-1,$$

and thus $B_1 = 0$. As $x_k = \cos(k\pi/N)$ we have

$$x_{k+1} - x_k = O(1/n^2), \qquad k \leqslant 4 \text{ or } k \geqslant n-4. \tag{2.17}$$

Since

$$\eta_k(x) = O(1), \qquad k = 1, 2, ..., n,$$

we have $B_2 = O(1/n^2)$. From the relations

$$\sum_{k=4}^{n-3} |\varphi_k(x)| = O(1),$$

$$W_{k+1} - W_k = -4\sin^2\frac{\pi}{2N}(\cos\theta_{k+1} - \cos\theta_k) = O\left(\frac{1}{n^3}\right),$$

using (2.16), we obtain $B_3 = O(1/n^2)$. This proves $I_1 = O(1/n^2)$ and Lemma 3 is established.

3. Proof of the Theorem

Using the mean value theorem, we have

$$
\begin{aligned}
f(x_k) - f(x) = f'(x + a_k(x_k - x))(x_k - x), \\
0 \leqslant a_k \leqslant 1,\ k = 1, 2, ..., n;
\end{aligned} \tag{3.1}
$$

$$f(x_k)-f(x_{k-1})=f'(t_k)(x_k-x_{k-1}),$$
$$x_k\leqslant t_k\leqslant x_{k-1},\ k=2,3,...,n. \tag{3.2}$$

Let us denote

$$h_1=\sum_{k=1,2,n-1,n} e_k(f(x_k)-f(x))\,m_k(x)$$
$$+\sum_{k=3,n-1}(f(x_k)-f(x))(m_k(x)+m_{k-1}(x)),$$
$$e_1=e_n=4,\ e_2=e_{n-1}=2;$$

$$h_2=\sum_{k=3,n-1} f'(t_k)(x_{k-1}-x_k)\,m_{k-1}(x)$$
$$+\sum_{k=2,3}(f'(t_k)-f'(x))(x_k-x_{k+1})\,m_k(x)$$
$$+\sum_{k=n-2,n-1}(f'(t_k)-f'(x))(x_k-x_{k-1})\,m_k(x);$$
$$h_3=f'(x)\sum_{k=3}^{n-2}\sum_{j=0}^{2}\binom{2}{j}(x_{k+j-1}-x)\,m_{k+j-1}(x);$$
$$h_4=\sum_{k=3}^{n-2}[(f'(x+\alpha_k(x_k-x))-f'(x))(x_k-x)\,p_k(x)+(f'(t_k)$$
$$-f'(t_{k-1}))(x_{k-1}-x_k)\,m_{k-1}(x)];$$
$$h_5=\sum_{k=4}^{n-3}(f'(t_k)-f'(x))(-x_{k+1}+2x_k-x_{k-1})\,m_k(x).$$

It is easy to see that

$$\sum_{k=1}^{n} m_k(x)=1.$$

From (1.4) we have

$$4(F_n(f,x)-f(x))=4\sum_{k=1}^{n}(f(x_k)-f(x))\,m_k(x)=\sum_{j=1}^{5}h_j.$$

Using (2.17) and Lemma 3, we have

$$h_2=O\left(\frac{\|f'\|}{n^2}\right),\qquad h_3=O\left(\frac{\|f'\|}{n^2}\right). \tag{3.3}$$

Using (2.7), (2.12), (2.17) and

$$1-x_k^2=1-\cos^2\theta_k=\sin^2\frac{k\pi}{N}=O\left(\frac{1}{n^2}\right), \qquad k\leqslant 4,$$

we obtain

$$\begin{aligned}
&\sum_{k=1,2} e_k(f(x_k)-f(x))\,m_k(x)\\
&\quad=\sum_{k=1,2} e_k f'(\gamma_k)(x_k-x)\,m_k(x)\\
&\quad=f'(\gamma_1)[3(x_1-x)\,\mu_1(x)+(x_2-x)\,\mu_2(x)+(x_1-x_2)\,\mu_2(x)]\\
&\qquad+\tfrac{1}{2}f'(\gamma_2)[2(x_2-x)\,\mu_2(x)+(x_1-x)\,\mu_1(x)+(x_3-x)\,\mu)\,\mu_3(x)\\
&\qquad+(x_2-x_1)\mu_1(x)+(x_2-x_3)\,\mu_3(x)]\\
&\quad=O\left(\frac{\|f'\|}{n^2}\right).
\end{aligned}$$

Similarly, we have

$$\sum_{k=n-1,n} e_k(f(x_k)-f(x))\,m_k(x)=O\left(\frac{\|f'\|}{n^2}\right),$$

$$\sum_{k=3,n-1}(f(x_k)-f(x))(m_k(x)+m_{k-1}(x))=O\left(\frac{\|f'\|}{n^2}\right),$$

and thus

$$h_1=O\left(\frac{\|f'\|}{n^2}\right). \tag{3.4}$$

To estimate h_4 and h_5, let x_j be the nearest node to $x\in[-1,1]$. Then

$$h_4=\left\{\sum_{k=j-3}^{j+3}+\sum_{\substack{k=3\\ k\neq j-3,\dots,j+3}}^{n-2}\right\}(\cdots)=E_1+E_2.$$

As $x=\cos\theta$, $x_k=\cos\theta_k$, we have

$$|x_k-x|=O(|\theta-\theta_k|), \qquad |x_k-x_{k-1}|=O(1/n). \tag{3.5}$$

Using (2.1), (2.4), and (3.5) we get

$$E_1=O(1)\frac{1}{n}\,\omega\left(f',\frac{\sqrt{1-x^2}}{n}+\frac{1}{n^2}\right).$$

By (2.2)-(2.6), (3.5) and as $\theta_k = k\pi/N$ and

$$\omega(f', k\delta) \leqslant (k+1)\,\omega(f', \delta),$$

we have

$$E_2 = O(1)\sum\left[\omega\left(f', \frac{|k-j|\sqrt{1-x^2}}{n} + \frac{(k-j)^2}{n^2}\right)\cdot\frac{|k-j|}{n}\cdot\left(\frac{1}{|k-j|^5}+\frac{1}{n^4}\right)\right.$$
$$\left. + \omega\left(f', \frac{\sqrt{1-x^2}}{n} + \frac{|k-j|}{n^2}\right)\cdot\frac{1}{n}\cdot\left(\frac{1}{|k-j|^3}+\frac{1}{n^2}\right)\right]$$
$$= O(1)\frac{1}{n}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right).$$

Therefore

$$h_4 = O(1)\frac{1}{n}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right). \tag{3.6}$$

In the same way, using (2.1)–(2.2), (2.4)–(2.5), and (2.16), we have

$$h_5 = \left\{\sum_{k=j-2}^{j+2} + \sum_{\substack{k=4 \\ k\neq j-2,\dots,j+2}}^{n-3}\right\}(\cdots)$$
$$= O(1)\frac{1}{n^2}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)$$
$$+ O(1)\sum\omega\left(f', \frac{|k-j|\sqrt{1-x^2}}{n} + \frac{(k-j)^2}{n^2}\right)\cdot\frac{1}{n^2}\cdot\left(\frac{1}{|k-j|^3}+\frac{1}{n^2}\right)$$
$$= O(1)\frac{1}{n}\,\omega\left(f', \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \tag{3.7}$$

and the theorem is proved.

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